PROCEEDINGS

OF

THE ROYAL IRISH ACADEMY.

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November 30th, 1846. (Stated Meeting.)

REV. HUMPHREY LLOYD, D. D., President, in the Chair.

Thomas Moore, Esq., having been specially recommended by the Council, was elected an Honorary Member.

The Rev. William Roberts, F. T. C. D., read a paper on the definite integral

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 + n \sin^2\phi)}{\sqrt{(1 - k^2 \sin^2\phi)}} d\phi.$$

It is clear that the only admissible values of n (real) are those of the parameter of an elliptic function of the third kind, to the modulus k, namely, $\cot^2\theta$, $-1 + k'^2 \sin^2\theta$, and $-k^2\sin^2\theta$, where k' is the complement of k. The value of the definite integral may, in each of these cases, be expressed by elliptic functions of the first and second kinds, and by the remarkable transcendant $\Upsilon \left(\int \frac{\mathrm{E}(\phi)\, d\phi}{\sqrt{(1 - k^2 \sin^2\phi)}}\right)$ by the aid of which functions of the third species, with a logarithmic parameter, can be

calculated by tables of double entry. In fact, we have the following formulæ,

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 + \cot^2\theta \sin^2\phi)}{\sqrt{(1 - k^2 \sin^2\phi)}} d\phi = \tag{1}$$

$$\pi F(k', \theta) - 2F(k)\Upsilon(k', \theta) - \{E(k) - F(k)\}\{F(k', \theta)\}^2$$
$$- \tfrac{1}{2}\pi F(k') - F(k)\log(k\sin^2\theta)$$

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 - (1 - k'^2 \sin^2\theta)\sin^2\phi)}{\sqrt{(1 - k^2 \sin^2\phi)}} d\phi = \tag{2}$$

$$\pi F(k', \theta) - 2F(k)\Upsilon(k', \theta) - \{E(k) - F(k)\}\{F(k', \theta)\}^2$$
$$- \tfrac{1}{2}\pi F(k') + \log\left(\frac{k'}{k}\right) F(k).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 - k^2 \sin^2\theta \sin^2\phi)}{\sqrt{(1 - k^2 \sin^2\phi)}} d\phi =$$
$$E(k)\{F(k, \theta)\}^2 - 2F(k)\Upsilon(k, \theta). \tag{3}$$

In equation (3) if we put $\theta = \frac{1}{2}\pi$, we will have, recollecting that

$$\Upsilon(\tfrac{1}{2}\pi) = \tfrac{1}{2}F(k)E(k) - \tfrac{1}{2}\log k',$$

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 - k^2 \sin^2\phi)}{\sqrt{(1 - k^2 \sin^2\phi)}} d\phi = \log(k') F(k). \tag{4}$$

Again, $\theta_{,}$ being the amplitude of the semi-complete function, we have

$$\sin^2\theta_{,} = \frac{1}{1 + k'}$$

and,

$$\Upsilon(\theta_{,}) = \tfrac{1}{8} F(k) E(k) - \tfrac{1}{4}\log\left(\frac{2k' \sqrt{k'}}{1 + k'}\right);$$

so that

$$\int_0^{\frac{1}{2}\pi} \frac{\log(\cos^2\phi + k'\sin^2\phi)}{\sqrt{(1 - k^2 \sin^2\phi)}} d\phi = \tfrac{1}{2}\log\left(\frac{2k'\sqrt{k'}}{1 + k}\right) F(k). \tag{5}$$

The values of the definite integrals (4) and (5) have been

already deduced by Mr. Roberts from entirely different considerations, and published in Liouville's *Journal de Mathematiques*, May, 1846.

Some other interesting results may be obtained from our general formulæ. Thus, if in (2) we put $\theta = 0$, we will have

$$\int_0^{\frac{1}{2}\pi} \frac{\log(\cos\phi)\, d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = \tfrac{1}{2}\log\left(\frac{k'}{k}\right) \mathrm{F}(k') - \tfrac{1}{4}\pi \mathrm{F}(k'), \qquad (6)$$

from which we may deduce, by an easy transformation,

$$\int_0^{\frac{1}{2}\pi} \frac{\log(\sin\phi)\, d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = \tfrac{1}{2}\log\left(\frac{1}{k}\right) \mathrm{F}(k) - \tfrac{1}{4}\pi \mathrm{F}(k')\,; \qquad (7)$$

and, consequently,

$$\int_0^{\frac{1}{2}\pi} \frac{\log(\tan\phi)\, d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = \tfrac{1}{2}\log\left(\frac{1}{k'}\right) \mathrm{F}(k). \qquad (8)$$

If we suppose k to vanish in formulæ (6) and (7), we obtain the well-known results, originally given by Euler,

$$\int_0^{\frac{1}{2}\pi} \log(\cos\phi)\, d\phi = \int_0^{\frac{1}{2}\pi} \log(\sin\phi)\, d\phi = \tfrac{1}{2}\pi \log \tfrac{1}{2}.$$

Denoting $\sqrt{(1 - k^2\sin^2\phi)}$ by Δ, we can also derive from the above the value of the definite integral

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 \pm \Delta\sin\theta)}{\Delta}\, d\phi. \qquad (9)$$

For, the sum of the integrals

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1 + \Delta\sin\theta)\, d\phi}{\Delta} \text{ and } \int_0^{\frac{1}{2}\pi} \frac{\log(1 - \Delta\sin\theta)}{\Delta}\, d\phi$$

may be found from (1), and their difference from the formula

$$\int_0^{\frac{1}{2}\pi} \log\left\{\frac{1 + \Delta\sin\theta}{1 - \Delta\sin\theta}\right\} \frac{d\phi}{\Delta} = \pi \mathrm{F}(k', \theta)$$

which Mr. Roberts has demonstrated in the *Journal de Mathematiques*, May, 1846.

In conclusion it may be observed, that the particular results, (4), (6), (7), (8), are nothing more than immediate consequences of Mr. Jacobi's factorial developments of the trigonometrical functions of the amplitude of an elliptic function, in terms of the function itself.—*Traité des Fonctions Elliptiques,* tom. iii. page 97. It may be seen that they follow at once from these expansions, if we remember that

$$\int_0^\pi \log(1 \pm 2a\cos x + a^2)\, dx = 0$$

when a is less than unity; a theorem proved by Poisson in the seventeenth *cahier* of the *Journal de l'Ecole Polytechnique.*

Sir William R. Hamilton stated the following theorems of central forces, which he had proved by his calculus of quaternions, but which, as he remarked, might be also deduced from principles more elementary.

If a body be attracted to a fixed point, with a force which varies directly as the distance from that point, and inversely as the cube of the distance from a fixed plane, the body will describe a conic section, of which the plane intersects the fixed plane in a straight line, which is the polar of the fixed point with respect to the conic section.

And in like manner, if a material point be obliged to remain upon the surface of a given sphere, and be acted on by a force, of which the tangential component is constantly directed (along the surface) towards a fixed point or pole upon that surface, and varies directly as the sine of the arcual distance from that pole, and inversely as the cube of the sine of the arcual distance from a fixed great circle; then the material point will describe a spherical conic, with respect to which the fixed great circle will be the polar of the fixed point.

Thus, a spherical conic would be described by a heavy point upon a sphere, if the vertical accelerating force were to